MINIMAL MODELS OF THEORIES OF ONE FUNCTION SYMBOL[†]

BY

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ABSTRACT

If T_1 is a theory of one function symbol and T_1 has a minimal model which is not prime, then T_1 has $2^{\aleph_0}$ non-isomorphic minimal models.

Introduction

A model is *minimal* if it has no proper elementary submodels. An open question is: What can be the number of non-isomorphic minimal models of a first order countable theory? Let L_1 be the language containing just a one-place function symbol f and equality. The purpose of this paper is to prove the following theorem, answering a question of Shelah.

MAIN THEOREM. *If T_1 is a theory in L_1, and T_1 has a minimal model which is not prime, then T_1 has $2^{\aleph_0}$ non-isomorphic minimal models.*

Actually we prove the same conclusion for theories T in a language containing a binary predicate $R(x, y)$ where $T \vdash \forall x \exists^{\leq 1} y\, R(x, y)$. The crux of the proof lies in showing that minimal models of T_1 are simple enough so that the theorem is implied by the following general principle.

If T (in any first-order language) has a non-prime minimal model, and the class of minimal models of T includes all models omitting some type, then T has $2^{\aleph_0}$ non-isomorphic minimal models. See Theorem 1.4 and Example 1.5.

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Notation and Definitions

Models are denoted by M, N, etc;. we denote the model and its universe set by the same letter. Thus we write both $\bar{a} \in M$ (meaning that $\bar{a} = \langle a_1, \cdots, a_k \rangle$ is a sequence whose members are elements of M) and also $M \models \phi(\bar{a})$ (meaning that the sequence $\bar{a}$ satisfies the formula $\phi(\bar{x})$ in the model M). If $\bar{a} = \langle a_1, \cdots, a_k \rangle$, $\bar{b} = \langle b_1, \cdots, b_l \rangle$, then $\bar{a} \wedge \bar{b} = \langle a_1, \cdots, a_k, b_1, \cdots, b_l \rangle$. We assume the reader to be familiar with Vaught [4] where some of the following definitions are to be found. M is a prime model of T if M is elementarily embeddable in every model of T. A type $p = p(v) = \{\phi_i(v) : i < \omega\}$ is atomic (in T) if there is some formula $\phi(v)$ such that $T \vdash \exists v\, \phi(v)$ and $T \vdash \forall v\, (\phi(v) \to \phi_i(v))$ for all $i < \omega$; $\phi(v)$ is called an atom. p is a complete type if for all $\psi(v)$, p implies $\psi(v)$ or p implies $\neg\, \psi(v)$. $\psi(v)$ is atomless if there is no atom $\phi(v)$ such that $T \vdash \forall v\, (\phi(v) \to \psi(v))$. $\exists^{\leqq n} x\, \phi(x)$ means that there are $\leqq n$ x such that $\phi(x)$; $M \models \exists^{<\omega} x\, \phi(x)$ means that there is $n < \omega$ such that $M \models \exists^{\leqq n} x \phi(x)$. The element a is algebraic over the set B in M if there is a formula $\phi(x, \bar{y})$ and $\bar{b} \in B$ such that $M \models \exists^{<\omega} x\, \phi(x, \bar{b}) \wedge \phi(a, \bar{b})$. All k, l, m, n, q, r designate natural numbers (except in Lemma 1.3 where q designates a type).

1.

In this section we shall be dealing with general model theory, not necessarily models of L_1. We use the following theorem of Engeler and Ryll-Nardzewski (see Chang [2]).

THEOREM 1.1. *Let T be a complete and countable theory, and let p be a type. Then the following two conditions are equivalent.*

(i) *T has a model omitting p.*

(ii) *For every formula $\phi(x)$ in the language of T there is a formula $\pi(x) \in p$ such that $T \vdash \exists x\, \phi(x) \to \exists x(\phi(x) \wedge \neg\, \pi(x))$.*

The following theorem is from Vaught [4].

THEOREM 1.2. (i) *All of the prime models of a countable theory T are isomorphic.*

(ii) *If T has no prime model, then there is an atomless formula.*

From this it is clear that if T has a prime minimal model M, then every minimal model of T is isomorphic to M, and if T has a prime model which is not minimal, then T has no minimal models.

LEMMA 1.3. *If there is an atomless formula in T and there is a model omitting the type p, then there are $2^{\aleph_0}$ non-isomorphic countable models omitting p.*

PROOF. The proof is actually well known, but for the sake of completeness we include a sketch. Assume that M is a model of T which omits p. Let $\psi(x)$ be an atomless formula. It is sufficient to show that there are $2^{\aleph_0}$ types which can be realized in models of T omitting p. It is clear that if q is a complete type then $T_q = T \cup \{\chi(c): \chi(v) \in q\}$ is a complete theory, where c is a new constant. Thus by Theorem 1.1 it is sufficient to show that there are $2^{\aleph_0}$ complete types q such that for all $\phi(x,v)$ in the language of T there is $\pi(x) \in p$ such that $T_q \vdash \exists x\ \phi(x,c) \rightarrow \exists x(\phi(x,c) \wedge \neg \pi(x))$, that is, if $\exists x\,\phi(x,v)$ follows from q then so does $\exists x(\phi(x,v) \wedge \neg \pi(x))$ for some $\pi \in p$. Now we construct by induction on n formulas $\psi_{\langle \varepsilon_0,\cdots,\varepsilon_{n-1}\rangle}(v)$, where $\varepsilon_i \in \{0,1\}$, such that $\psi_{\langle\rangle} = \psi$, $\psi_{\langle \varepsilon_0,\cdots,\varepsilon_{n-1}\rangle} \rightarrow \psi_{\langle \varepsilon_0,\cdots,\varepsilon_i\rangle}$ for $i < n$, and the following holds:

(i) $T \vdash (\exists v_0)\psi_{\langle \varepsilon_0,\cdots,\varepsilon_{i-1},0\rangle}(v_0) \wedge (\exists v_1)\psi_{\langle \varepsilon_0,\cdots\, \varepsilon_{n-1},1\rangle}(v_1) \wedge (\neg\exists v)(\psi_{\langle \varepsilon_0,\cdots,\varepsilon_{n-1},0\rangle}(v) \wedge \psi_{\langle \varepsilon_n,\cdots\, \varepsilon_{n-1},1\rangle}(v))$.

(ii) If $T \vdash \psi_{\langle \varepsilon_0,\cdots,\varepsilon_{n-1}\rangle}(v) \rightarrow \exists x\ \phi(x,v)$ then there is $\pi \in p$ such that $T \vdash \psi_{\langle \varepsilon_0,\cdots,\varepsilon_m\rangle}(v) \rightarrow \exists x(\phi(x,v) \wedge \neg \pi(x))$ for some $m \geqq n$.

(iii) For all $\phi(v)$ there is some n such that $T \vdash \psi_{\langle \varepsilon_0,\cdots,\varepsilon_{n-1}\rangle}(v) \rightarrow \phi(v)$ or

$$T \vdash \psi_{\langle \varepsilon_0,\cdots\, \varepsilon_{n-1}\rangle}(v) \rightarrow \neg\, \phi(v).$$

We can guarantee that (i) will hold since $\psi(v)$ is atomless. (ii) is realized by using the fact that there are at most $\aleph_0$ ϕ's such that $T \vdash \psi_{\langle \varepsilon_0,\cdots,\varepsilon_{n-1}\rangle}(v) \rightarrow \exists x\,\phi(x,v)$; for each of them, since $M \vDash T$ and M omits p, there is $x \in M$ satisfying $\phi(x,v)$ which does not realize p. So we can find $\pi \in p$ such that $M \vDash \neg \pi(x)$. (iii) is easily accomplished. Now, given $\eta = \langle \varepsilon_i: i < \omega\rangle$, put $q_\eta = \{\psi_{\langle \varepsilon_0,\cdots,\varepsilon_{n-1}\rangle}(v): n < \omega\}$. Then q_η satisfies our requirements and there are $2^{\aleph_0}$ such η's.

THEOREM 1.4. *If the class of minimal models of T contains all models of T omitting some type, and T has a non-prime minimal model, then T has $2^{\aleph_0}$ non-isomorphic minimal models.*

PROOF. Since T has a non-prime minimal model, T does not have a prime model. Thus T has an atomless formula. By the lemma, T has $\geqq 2^{\aleph_0}$ non-isomorphic minimal models, so clearly the number is $2^{\aleph_0}$.

REMARK. What we have done so far for types of one variable immediately generalizes to types of finite sequences $p(v_1,\cdots,v_n) = \{\phi(v_1,\cdots,v_n)\}_{\phi\varepsilon p}$.

EXAMPLE 1.5. A special case of the above theorem is when T has a non-prime minimal model M in which every element is algebraic over every other. Then M omits the type $p(v_1, v_2) = \{\exists^{\leq n} x\, \theta(v_1, x) \to \neg\, \theta(v_1, v_2) : \theta \in L,\, n < \omega\}$. It is not difficult to verify that every model omitting p is minimal. Thus by Theorem 1.4, T has $2^{\aleph_0}$ non-isomorphic minimal models.

2.

From now on we shall be dealing with models and theories of L_1. Actually, the language has one binary predicate $R(x, y)$ and we add the axiom $\forall x\, \exists^{\leq 1} y\, R(x, y)$. It is convenient in many cases to write $R(x, y)$ as $f(x) = y$. We include the possibility that f is not always defined, that is, $M \models \neg\, \exists y\; R(x, y)$ for some $x \in M$.

An expression like $f^m(x_0) = f^n(y_0)$ will be an abbreviation for

$$\exists x_1, \cdots, x_m\, \exists y_1, \cdots, y_n (\bigwedge_{i<m} R(x_i, x_{i+1}) \wedge \bigwedge_{j<n} R(y_j, y_{j+1}) \wedge x_m = y_n).$$

DEFINITION 2.1. The distance between a and b relative to a set A is $d_A(a, b) = \min\{r$: there are k, l such that $k + l = r$ and there are $x_0, \cdots, x_k, y_0, \cdots, y_l \in A$ such that $a = x_0$, $b = y_0$, $f(x_i) = x_{i+1}$ for $i < k$, $f(y_j) = y_{j+1}$ for $j < l$, and $x_k = y_l\}$.

If no such r exists write $d_A(a, b) = \infty$. A path from a to b is a sequence $\langle x_0, \cdots, y_l \rangle$ as above.

Note that if A contains no loops (that is, $n \geqq 1 \Rightarrow A \models f^n(x) \neq x$) then there is at most one path from a to b; in any case $d_A(a, b)$ defines a metric. If $A \subseteq B$ then $d_B(a, b) \leqq d_A(a, b)$.

Any model breaks up into a disjoint union of components, a component being the set of points of finite distance from a given point. If $a \in B$ let $Nbd^r_B a$ be the set of points in B of distance $\leqq r$ from a. If A is a set let $Nbd^r_B A = \bigcup_{a \in A} Nbd^r_B a$. $Nbd^r_B A$ is called the r-neighborhood of A in B.

DEFINITION 2.2. Let A be a set, $a_i \in A$, for $i = 1, \cdots, n$. The q-type of $\langle a_1, \cdots, a_n \rangle$ over A is $\{\psi(x_1, \cdots, x_n) : \psi(x_1, \cdots, x_n)$ has $\leqq q$ quantifiers and $A \models \psi(a_1, \cdots, a_n)\}$.

Clearly every q-type is equivalent to a single formula, and there are only a finite number of non-equivalent q-types (for fixed q).

Write $Nbd^r_A \bar{a} \equiv_q Nbd^r_B \bar{b}$ if the q-type of $\bar{a}$ over $Nbd^r_A \bar{a}$ is the same as the q-type of $\bar{b}$ over $Nbd^r_B \bar{b}$.

LEMMA 2.0. (i) *For any n, q, r there is a number $q^* = q^*(n, q, r)$ such that if $\psi(\bar{x}) = \psi(x_1, \cdots, x_n)$ is a formula with n free variables and q quantifiers, then there is a formula $\psi^*(\bar{x})$ with the same free variables and q^* quantifiers such that for all sets A and $\bar{a} \in A$, $Nbd^r_A \bar{a} \models \psi(\bar{a}) \Leftrightarrow A \models \psi^*(\bar{a})$.*

(ii) *If $Nbd^r_A \bar{a} \equiv_q Nbd^r_A \bar{b}$, $Nbd^r_A \bar{c} \equiv_q Nbd^r_A \bar{d}$, and $Nbd^r_A \bar{a} \cap Nbd^r_A \bar{c} = Nbd^r_A \bar{b} \cap Nbd^r_A \bar{d} = \varnothing$, then $Nbd^r_A \bar{a} {}^\wedge \bar{c} \equiv_q Nbd_A{}^r \bar{b} {}^\wedge \bar{d}$.*

(iii) *If $Nbd^r_A a_0 \equiv_q Nbd^r_A a^*_0$, $f^n(a_0) = f^n(a^*_0) = b$ for some n, neither a_0 nor a^*_0 are in loops, and $\bar{c} = \langle c_1, \cdots, c_k \rangle$ is such that the only paths leading from a_0 or a^*_0 to any c_i go through b, then $Nbd_A \langle a_0 \rangle {}^\wedge \bar{c} \equiv_q Nbd^r_A \langle a^*_0 \rangle {}^\wedge \bar{c}$.*

PROOF. (i) is clear from the definition of $Nbd^r_A \bar{a}$. (ii) and (iii) follow by using Ehrenfeucht's game criterion, for example. See [3].

I am indebted to M. Rubin for a discussion which led to the statement of the following lemma.

LEMMA 2.1. *For any m and n there are numbers $r(m, n)$ and $q(m, n)$ such that for any $\psi(\bar{x})$ with n free variables and m quantifiers, and for any model M and every $\bar{a}, \bar{b}$ from M of length n, if*

$$Nbd^{r(m,n)}_M \bar{a} \equiv_{q(m,n)} Nbd^{r(m,n)}_M \bar{b}$$

then $M \models \psi(\bar{a}) \leftrightarrow \psi(\bar{b})$.

REMARK. Roughly speaking, the lemma says that the satisfaction of $\psi(\bar{x})$ in M is determined by the q-type of $\bar{x}$ over its r-neighborhood in M where q and r depend only on the length of $\bar{x}$ and the number of quantifiers in ψ. The proof we give is combinatorial (in that it does not use the compactness theorem, for example) and gives estimates for q and r. Gaifman has given an alternative model-theoretic proof of the lemma. Our main task is to show that if

$$Nbd^{r(m+1,n)} \bar{a} \equiv_{q(m+1,n)} Nbd^{r(m+1,n)} \bar{b}$$

then for all c there is d such that $Nbd^{r(m,n+1)} \bar{a} {}^\wedge \langle c \rangle \equiv_{q(m,n+1)} Nbd^{r(m,n+1)} \bar{b} {}^\wedge \langle d \rangle$ (throughout the proof we write Nbd in place of Nbd_M). This is done by cases according to the location of c relative to $\bar{a}$ and $\bar{b}$.

PROOF. We define $r(m, n)$ by induction as follows: $r(0, n) = 1$ and $r(m + 1, n) = 8n\, r(m, n + 1)$. As for $q(m, n)$, in the course of the proof we shall just show that we can choose it large enough, and then at the end of the proof state a numerical value which allows all the stages of the proof to work.

The proof is by induction on m for all n. If $m = 0$ the claim is true since a formula with no quantifiers just says which of its free variables are equal to, or images of, the others. Also if the claim is true for $\psi(\bar{x})$ then it is true for $\neg\psi(\bar{x})$. So we must show that if the claim is true for m and n, and if $\exists v\, \psi(v, x_1, \cdots, x_n)$ is a formula with $m + 1$ quantifiers and n free variables and

$$Nbd^{r(m+1,n)}\bar{a} \equiv_{q(m+1,n)} Nbd^{r(m+1,n)}\bar{b}$$

then $M \models \exists v\psi(v, \bar{a}) \leftrightarrow \exists v\, \psi(v, \bar{b})$. Assume $M \models \exists v\, \psi(v, \bar{a})$. Then there is c such that $M \models \psi(c, \bar{a})$. Now $\psi(v, \bar{x})$ is a formula with m quantifiers and $n + 1$ free variables so by the induction hypothesis it is sufficient to find d such that

$$Nbd^{r(m,n+1)}\bar{a}^\frown \langle c\rangle \equiv_{q(m,n+1)} Nbd^{r(m,n+1)}\bar{b}^\frown \langle d\rangle.$$

There are three cases.

Case 1. There is i $(1 \leqq i \leqq n)$ such that $Nbd^{r(m,n+1)}c \subseteq Nbd^{r(m+1,n)}a_i$. Let $\phi(x, \bar{y})$ be a formula equivalent to the $q(m, n + 1)$-type of $\bar{a}^\frown\langle c\rangle$ over $Nbd^{r(m+1,n)}\bar{a}$. Thus $Nbd^{r(m+1,n)}\bar{a} \models \exists x\ \phi(x, \bar{a})$ (c is such an x). If $q(m + 1, n)$ is chosen large enough so that $q(m + 1, n) \geqq$ the number of quantifiers in $\exists x\ \phi(x, \bar{y})$ then by the $q(m + 1, n)$-equivalence of $Nbd^{r(m+1,n)}\bar{a}$ and $Nbd^{r(m+1,n)}\bar{b}$ we obtain $Nbd^{r(m+1,n)}\bar{b} \models \exists x\, \phi(x, \bar{b})$. Now take d satisfying $Nbd^{r(m+1,n)}\bar{b} \models \phi(d, \bar{b})$; clearly $Nbd^{r(m,n+1)}\bar{a}^\frown\langle c\rangle \equiv_{q(m,n+1)} Nbd^{r(m,n+1)}\ \bar{b}^\frown \langle d\rangle$ as was to be proved.

Case 2. $Nbd^{r(m,n+1)}c \cap Nbd^{r(m,n+1)}\bar{a}^\frown \bar{b} = \varnothing$. Here we may take $d = c$ by Lemma 2.0 (ii).

Case 3. Cases 1 and 2 do not happen. Thus for all i

(0) $Nbd^{r(m,n+1)}c \nsubseteq Nbd^{r(m+1,n)}a_i$, but $Nbd^{r(m,n+1)}c \cap Nbd^{r(m,n+1)}\bar{a}^\frown \bar{b} \neq \varnothing$. If for some i $Nbd^{r(m,n+1)}c \cap Nbd^{r(m,n+1)}a_i \neq \varnothing$, then $Nbd^{r(m,n+1)}c \subseteq Nbd^{r(m+1,n)}a_i$, contradiction. (Actually, here, we only need the fact that $r(m + 1, n) \geqq 3\, r(m, n+1)$).

Thus we may assume that $Nbd^{r(m,n+1)}c$ intersects an $r(m, n + 1)$-neighborhood of some element of $\bar{b}$, and without loss of generality,

(1) $Nbd^{r(m,n+1)}c \cap Nbd^{r(m,n+1)}b_1 \neq \varnothing$.

Put $r(m, n + 1) = t$, $r(m + 1, n) = r$ then $r = 8nt$. Since $Nbd^t c \cap Nbd^t a_i = \varnothing$ for all i, it is sufficient to find d with $Nbd^t c \equiv_{q(m,n+1)} Nbd^t d$ and

(2) $Nbd^t d \cap Nbd^t b_i = \varnothing$ for all i (this by Lemma 2.0(ii)).

From (1) follows, as above in Case 1,

(3) $Nbd^t c \subseteq Nbd^{8nt} b_1$ and so we can find d_1 such that

(4) $Nbd^t\langle a_1, d_1\rangle \equiv_{q(m,n+1)} Nbd^t\langle b_1, c\rangle$. If (2) holds for $d = d_1$ we are through;

if not, $Nbd^t d_1 \cap Nbd^t b_i \neq \emptyset$ for some i. We shall now show that $b_i \neq b_1$. For assume that $Nbd^t_{d_1} d_1 \cap Nbd^t b_1 \neq \emptyset$. Then $d(b_1, d_1) \leqq 2t$. By (1), $d(b_1, c) \leqq 2t$ so by taking q large enough so that each of the formulas which assert that the distance between r points is k, $k \leqq r$, has less than q quantifiers, we obtain from (4) that also $d(a_1, d_1) \leqq 2t$. Thus $d(a_1, b_1) \leqq 4t$. On the other hand $d(b_1, c) \leqq 2t$ and $d(a_1, c) > 8nt - t$ by (0). So $d(a_1, b_1) > 8nt - 3t \geqq 5t$, contradiction. Thus $b_i \neq b_1$. So assume

(5) $Nbd^t d_1 \cap Nbd^t b_2 \neq \emptyset$. As above we can find d_2 such that

(6) $Nbd^t\langle a_2, d_2\rangle \equiv_{q(m,n+1)} Nbd^t\langle b_2, d_1\rangle$. If (2) holds for $d = d_2$ we are through; if not, $Nbd^t d_2 \cap Nbd^t b_i \neq \emptyset$ for some i. We shall show that $b_i \neq b_1$, b_2. First, $Nbd^t d_2 \cap Nbd^t b_1 = \emptyset$: If not, then $d(b_1, d_2) \leqq 2t$, $d(a_2, d_2) \leqq 2t$ (by (5) and (6)) and so $d(b_1, a_2) \leqq 4t$. But $d(a_2, c) \geqq 8nt - t$ by (0) and $d(b_1, c) \leqq 2t$, so $d(b_1, a_2) \geqq 8nt - 3t \geqq 5t$, contradiction.

Now $Nbd^t d_2 \cap Nbd^t b_2 = \emptyset$. If not, $d(d_2, b_2) \leqq 2t$, $d(a_2, d_2) \leqq 2t$ and so $d(a_2, b_2) \leqq 4t$. Similarly, by (4), (5), $d(a_1, b_2) \leqq 4t$ and so $d(a_1, a_2) \leqq 8t$. Thus $d(b_1, b_2) \leqq 8t$. But then $d(b_1, d_2) \leqq 10t$, $d(a_2, b_1) \leqq 12t$, and $d(a_2, c) \leqq 14t$. But $d(a_2, c) \geqq 8nt - t \geqq 15t$ (since $n \geqq 2$). So $Nbd^t d_2 \cap Nbd^t b_2 = Nbd^t d_2 \cap Nbd^t b_1 = \emptyset$.

Continuing this way, after at most n steps we get d satisfying $Nbd^t c \equiv_{q(m,n+1)} Nbd^t d$ and (2). This completes the proof.

It can be seen that

$$q(m+1, n) = q^*(r(m, n+1), n+1, q(m, n+1)) \cdot NE(q^*(r(m, n+1), n+1, q(m, n+1)), n+1) + r(m+1, n) + 1$$

suffices, where q^* is from Lemma 2.0(i) and $NE(q, n)$ is the number of non-equivalent formulas having n free variables and q quantifiers.

Corollary 2.2. *Let M' be a component of M. For every formula $\psi(\bar{x})$ there is a formula $\psi'(\bar{x})$ such that for all $\bar{a} \in M'$,*

$$M' \vDash \psi(\bar{a}) \Leftrightarrow M \vDash \psi'(\bar{a}).$$

Proof. Assume $\psi(\bar{x})$ has n free variables and m quantifiers. By the previous lemma the satisfaction of $\psi(\bar{x})$ in M' depends only on the $q(m,n)$-type of $Nbd^{r/(m,n)}_{M'}\bar{x}$. There are only a finite number of non-equivalent $q(m,n)$-types. Assume that $\bar{a}_1, \cdots, \bar{a}_k \in M'$ such that $M' \vDash \psi(\bar{a}_i)$ for $1 \leqq i \leqq k$,

$$Nbd^{r(m,n)}_{M'}\bar{a}_i \not\equiv_{q(m,n)} Nbd^{r(m,n)}_{M'}\bar{a}_j$$

for all $1 \leqq i \neq j \leqq k$, and k is maximal. Let $\pi_i(\bar{x})$ be a formula equivalent to the $q(m,n)$-type of $\bar{a}_i$ over $Nbd_{M'}^{r(m,n)}$, and consider $\tilde{\psi}(\bar{x}) = \bigvee_{1 \leqq i \leqq k} \pi_i(\bar{x})$. Then for all $\bar{a} \in M'$, $M' \models \psi(\bar{a}) \Leftrightarrow Nbd_{M'}^{r(m,n)}\bar{a} \models \tilde{\psi}(\bar{a})$. Since M' is a component of M, $Nbd_{M'}^{r(m,n)}\bar{a} = Nbd_{M}^{r(m,n)}\bar{a}$ for all $\bar{a} \in M$. Thus $M' \models \psi(\bar{a}) \leftrightarrow Nbd_{M}^{r(m,n)}\bar{a} \models \tilde{\psi}(\bar{a})$. By Lemma 2.0(i) there is $\psi^*(\bar{a})$ such that $Nbd_{M}^{r(m,n)}\bar{a} \models \tilde{\psi}(\bar{a}) \Leftrightarrow M \models \psi^*(\bar{a})$. Thus $M' \models \psi(\bar{a}) \Leftrightarrow M \models \psi^*(\bar{a})$, as was to be proved.

COROLLARY 2.3. *If M' is a component of M, $N' \prec M'$, then $(M - M') \cup N' \prec M$. That is to say, by replacing a component by an elementary submodel of it one gets an elementary submodel of the whole model.*

PROOF. Let $\bar{a} \in (M - M') \cup N'$. We must show that if $M \models \exists x\ \psi(x, \bar{a})$ then there is $a_0 \in (M - M') \cup N'$ such that $M \models \psi(a_0, \bar{a})$. Let $\bar{a} = \bar{b}^\frown \bar{c}$ where $\bar{b} \in N'$, $\bar{c} \in M - M'$, and assume that $M \models \psi(a_0', \bar{b}, \bar{c})$ where $a_0' \in M'$. By Lemma 2.0(ii) and Lemma 2.1 it suffices to find $a_0 \in N'$ such that

$$Nbd_M^{r(m,n)}\langle a_0' \rangle ^\frown \bar{b} \equiv_{q(m,n)} Nbd_M^{r(m,n)}\langle a_0 \rangle ^\frown \bar{b}$$

(since the neighborhoods of a_0' and $\bar{b}$ are all disjoint from those of $\bar{c}$). This we may do since $N' \prec M'$.

COROLLARY 2.4. *If $f(a) = b$ and a is not algebraic over b, then*

$$M - \bigcup_{n < \omega} f^{-n}(a) \prec M$$

(where $f^{-0}(a) = \{a\}$).

PROOF. Assume $M \models \exists x\, \psi(x, \bar{c})$, $\bar{c} \in M - \bigcup_{n<\omega} f^{-n}(a)$. We must find such an x in $M - \bigcup_{n<\omega} f^{-n}(a)$. Notice that from the hypothesis of the corollary it follows that a cannot be in a loop (for then $a = f^k(b)$ where $k \geqq 0$ and a is definable from b). Let $M \models \psi(a_0, \bar{c})$ where $a_0 \in \bigcup_{n<\omega} f^{-n}(a)$, say $f^{n_0}(a_0) = a$. Take $q(m,n)$ and $r(m,n)$ corresponding to $\psi(x, \bar{y})$ from Lemma 2.1 and let $\phi(x_0)$ be a formula equivalent to the $q(m,n)$-type of a_0 over $Nbd_M^{r(m,n)}a_0$. Set $\phi_0(x_0, x_1) = (\phi^*(x_0) \wedge f^{n_0}(x_0) = x_1)$ and $\phi_1(x_1, x_2) = (\exists x_0 \phi_0(x_0, x_1) \wedge f(x_1) = x_2)$, where ϕ^* is taken from Lemma 2.0(i). Thus $M \models \phi_1(a, b)$. Since a is not algebraic over b, there are infinitely many $a_1 \in M$ satisfying $M \models \phi_1(a_1, b)$ and we can choose a^* such that $a^* \neq a$, $M \models \phi_1(a^*, b)$, a^* is not in a loop, and $(\bigcup_{n<\omega} f^{-n}(a^*)) \cap \bar{c} = \varnothing$. This is accomplished by first considering infinitely many a^* satisfying $M \models \phi_1(a^*, b)$ which are not in loops and then choosing one such that $(\bigcup_{n<\omega} f^{-n}(a^*)) \cap \bar{c} = \varnothing$, using the fact that if a_1^*, a_2^* are not in loops and $M \models \phi_1(a_1^*, b) \wedge \phi_1(a_2^*, b)$ then $\bigcup_{n<\omega} f^{-n}(a_1^*) \cap \bigcup_{n<\omega} f^{-n}(a_2^*) = \varnothing$ (since if $f^{m_1}(x) = a_1^*$, $f^{m_2}(x) = a_2^*$ then there would be a loop containing b and a_1^* or a_2^*). Thus $M \models \exists x_0 \phi_0(x_0, a^*) \wedge f(a^*) = b$. Let a_0^* be such that $M \models \phi_0(a_0^*, a^*)$.

We claim that a_0^* is the required x: First of all $a_0^* \in M - \bigcup_{n<\omega} f^{-n}(a)$ since $f^{n_0}(a_0^*) = a^*$ and $\bigcup_{n<\omega} f^{-n}(a) \cap \bigcup_{n<\omega} f^{-n}(a^*) = \emptyset$. Secondly we must show $M \models \psi(a_0^*, \bar{c})$. For this it suffices that $Nbd_M^{r(m,n)}\langle a_0 \rangle \frown \bar{c} \equiv_{q(m,n)} Nbd_M^{r(m,n)}\langle a_0^* \rangle \frown \bar{c}$. See Figure 1. What we do know is that $Nbd_M^{r(m,n)} a_0 \equiv_{q(m,n)} Nbd_M^{r(m,n)} a_0^*$, this by the

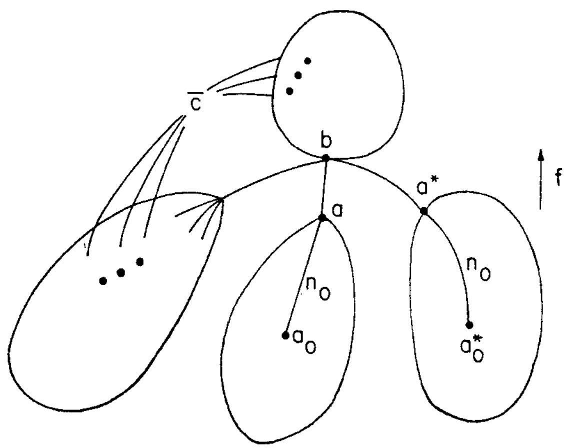

Fig. 1

choice of ϕ. But $a_0, a_0^*, b, \bar{c}$ satisfy the hypothesis of Lemma 2.0(iii) so actually $Nbd_M^{r(m,n)}\langle a_0 \rangle \wedge \bar{c} \equiv {}_q Nbd_M^{r(m,n)}\langle a_0^* \rangle \wedge \bar{c}$ as desired

3.

Now we shall deal with minimal models of L_1.

LEMMA 3.1. *If M consists of a single component and is a minimal model then for all $a, b \in M$, a is algebraic over b.*

PROOF. Since algebraicity is transitive (see, for example, Baldwin and Lachlan [1]) if a is not algebraic over b we may assume without loss of generality that $f(a) = b$. But then by Corollary 2.4, M has an elementary submodel not containing a, in contradiction to M's minimality.

THEOREM 3.2. (i) *If M consists of a single component and is a non-prime minimal model then there are $2^{\aleph_0}$ non-isomorphic minimal models elementarily equivalent to M.*

(ii) *If M is minimal and contains a component which is not a prime model in* *itself (that is, of its own theory) then there are $2^{\aleph_0}$ non-isomorphic minimal models elementarily equivalent to M.*

PROOF. (i) follows directly from Lemma 3.1 and Example 1.5.

(ii) If M' is the said component, then as in (i) we obtain $2^{\aleph_0}$ minimal models N' elementarily equivalent to M'. It is easy to see that $(M-M') \cup N' \equiv M$, $(M - M') \cup N'$ is minimal, and that there are $2^{\aleph_0}$ models of this form which are non-isomorphic.

We see from Theorem 3.2(ii) that in order to complete the proof of the Main Theorem we must show the following theorem.

THEOREM 3.3. *If M is minimal and not prime then M contains a component which is not prime.*

First we need a lemma.

LEMMA 3.4. *If M is minimal and M' is a component of M, then there is a formula $\phi^*(x)$, a possibly empty sequence $\bar{b} = \langle b_1, \cdots, b_n \rangle \in M - M'$, and a number r^* such that b_i and b_j are in different components of M for all $i \neq j$, and*

$$\emptyset \neq \{x : M \models \phi^*(x) \ \wedge \ \bigwedge_{i=1}^{n} d(x, b_i) > r^*\} \subseteq M'.$$

PROOF. Since M is minimal, we cannot obtain an elementary submodel by throwing out an entire component. So there is a formula $\psi(x, \bar{y})$ and a sequence $\bar{c} \in M - M'$ such that $M \models \exists x \ \psi(x, \bar{c})$ and the only such x is in M'. Also each component is a minimal model, so by Lemma 3.1 if c_i, c_j are in the same component then each is algebraic over the other. Thus we can find $\bar{b} = \langle b_1, \cdots, b_n \rangle \subseteq \bar{c}$ such that b_i and b_j are in different components and such that there is a formula $\tilde{\psi}(x, z)$ for which $M \models \forall x(\psi(x, \bar{c}) \equiv \tilde{\psi}(x, \bar{b}))$. Now let $M \models \tilde{\psi}(a, \bar{b})$; of course $a \in M'$. By Lemma 2.1 and Lemma 2.0(ii) if $Nbd^r_M a' \equiv_q Nbd^r_M a$ and $Nbd^r_M a' \cap Nbd^r_M \bar{b} = Nbd^r_M a \cap Nbd^r_M \bar{b} = \emptyset$ then $M \models \tilde{\psi}(a, \bar{b}) \equiv \tilde{\psi}(a', \bar{b})$. Take $\phi(x)$ to be a formula equivalent to the q-type of a over $Nbd^r_M a$, let $\phi^*(x)$ be the related formula from Lemma 2.0(i), and let $r^* = 2r$. Then $M \models \phi^*(a) \wedge \bigwedge_{i=1}^n d(a, b_i) > r^*$. On the other hand if $M \models \phi^*(a') \wedge \bigwedge_{i=1}^n d(a', b_i) > r^*$ then $Nbd^r_M a' \equiv_q Nbd^r_M a$ and $Nbd^r_M a' \cap Nbd^r_M \bar{b} = \emptyset$ so $M \models \tilde{\psi}(a', \bar{b})$. Thus $a' \in M'$.

Q.E.D.

Now we shall prove Theorem 3.3 in the following form.

THEOREM 3.5. *If M is minimal and each component is prime, then M is prime.*

PROOF. We must show that every sequence in M realizes an atomic type.

CLAIM. It is sufficient to show this for every sequence contained in a component of M.

PROOF OF THE CLAIM. Consider the special case where a_1, a_2 are in different components M_1 and M_2, and we know that $\psi_{a_1}(x)$, $\psi_{a_2}(x)$ isolate their types (in M). We must find $\psi_{\langle a_1,a_2\rangle}(x,y)$ which isolates the type of $\langle a_1,a_2\rangle$ in M. It is sufficient to find a formula $\psi_{\langle a_1,a_2\rangle}(x,y)$ which implies that x and a_1 have the same type, y and a_2 have the same type, and x,y are in different components.

Let $\bar{b}^1 = \langle b_1^1,\cdots,b_n^1\rangle$, $\bar{b}^2 = \langle b_1^2,\cdots,b_n^2\rangle$ be the sequences associated with M_1, M_2 respectively, from Lemma 3.4. Notice that without damaging the validity of Lemma 3.4 we may assume that the sequences are of the same length, $b_1^1\in M_2$, $b_1^2\in M_1$, and b_i^1, b_i^2 are in the same component for $i=2,\cdots,n$. Let $\phi_1^*(x)$, $\phi_2^*(x)$ be the formulas associated with M_1, M_2, respectively, from Lemma 3.4, r_1^* and r_2^* are the respective distances, and choose $a_i^*\in M_i$ satisfying $M\models\phi_i^*(a_i^*)$, $i=1,2$. Let

$$r > 8\cdot\max[\{r_1^*,r_2^*,d(a_1^*,b_1^2),d(a_2^*,b_1^1),d(a_1,a_1^*),d(a_2,a_2^*),\ d(b_2^1,b_2^2),\cdots,d(b_n^1,b_n^2)\}].$$

Notice that all of the above distances are finite. Define

$$\chi(x,y) = (\exists uv)\,[\phi_1^*(u)\wedge d(x,u) = d(a_1,a_1^*)\wedge\phi_2^*(v)\wedge d(y,v) = d(a_2,a_2^*)\wedge d(x,y)>r].$$

We claim that if $M\models\chi(x,y)$, then x and y must be in different components; the proof is similar to Lemma 2.1, Case 3. Assume by way of contradiction that $M\models\chi(x,y)$ but x and y are in the same component M'.

Let u,v be as in $\chi(x,y)$.

(1) If M' does not contain any b_i^1 or b_j^2 then clearly $d(u,b_i^1)>r_1^*$, $d(v,b_i^2)>r_2^*$ for all i. But since $\phi_1^*(u)$ and $\phi_2^*(v)$, u must be in M_1 and v must be in M_2 by Lemma 3.4, so x must be in M_1 and y must be in M_2, contradiction.

(2) If $b_{i_0}^1,b_{i_0}^2\in M'$ for some $i_0\geqq 2$ and $d(u,b_{i_0}^1)>r_1^*$ or $d(v,b_{i_0}^2)>r_2^*$ then either $u\in M_1$ or $v\in M_2$; so M' is M_1 or M_2, contradiction since $b_{i_0}^1,b_{i_0}^2\in M'$. So we may assume $d(u,b_{i_0}^1)\leqq r_1^*$ and $d(v,b_{i_0}^2)\leqq r_2^*$. But x is "close" to u (relative to r), u is close to $b_{i_0}^1$, $b_{i_0}^1$ is close to $b_{i_0}^2$, $b_{i_0}^2$ is close to v, and v is close to y. So x is close to y, in contradiction to $d(x,y)>r$.

(3) The last possibility is where M' is M_1 or M_2. The proof is similar.

Thus $M\models\chi(x,y)$ implies that x and y are in different components. All we have to do now is to take $\psi_{\langle a_1,a_2\rangle}(x,y) = \psi_{a_1}(x) \wedge \psi_{a_2}(y) \wedge \chi(x,y)$, and the *claim* is proved in the given special case. The proof of the general case is clear from here.

Now we continue with the proof of Theorem 3.5. Since M is minimal, each component is minimal and so in each component every element is algebraic over every other (Lemma 3.1). Thus by the *claim* it is sufficient to prove that each component contains some single element realizing an atomic type in M, for a sequence realizes an atomic type if each of its members is algebraic over an element realizing an atomic type.

Let M' be a component and let ϕ^*, r^* and $\bar{b} = \langle b_1, \cdots, b_n\rangle$ be as in Lemma 3.4. Choose $a \in M'$ such that $M \models \phi^*(a)$. There are two cases.

Case 1. There is no $x \in M'$ such that $M \models \phi^*(x)$ and $d(x,a) > r^*$. In this case we shall find an element $c \in M'$ which is algebraic (over $\emptyset$) in M, and hence realizes an atomic type.

a) There is $r < \omega$ such that $d(a, f^n(a)) \leqq r$ for all $n \geqq 0$; that is, either $f^n(a)$ is not defined for some n or M' contains a loop (note that a component can contain at most one loop). We shall deal with the case where M' contains a loop; the other case is treated similarly. Let $n \geqq 0$ be minimal such that $f^n(a)$ is in the loop, and let $c = f^n(a)$. Let $m > 0$ be minimal such that $f^m(c) = c$. Then $M \models \tau(c)$ where $\tau(x)$ is $f^m(x) = x \wedge \exists v(f^n(v) = x \wedge \phi^*(v))$. Also for any $x \in M'$, $M \models \tau(x)$ implies that x is in the loop in M'; so there are at most m elements in M' satisfying $M \models \tau(x)$. Now any other elements x of M satisfying $\tau(x)$ must be in the components of the b_i $(i = 1, \cdots, n)$, and clearly each such component can contain at most finitely such x. Thus $M \models \exists^{<\omega} x\, \tau(x)$, and c is algebraic in M.

b) $d(a, f^n(a)) = n$ for all $n \geqq 0$. Then clearly M' does not contain a loop and thus there is a unique path between every two elements of M'. Let $\pi(u,v)$ be the disjunction of all formulas of the form

$$(\exists z_1, \cdots, z_m)\left(\bigwedge_{1 \leqq i \neq j \leqq m} z_i \neq z_j \wedge f(u) = z_1 \wedge \rho_1 \wedge \cdots \wedge \rho_m\right)$$

where $m < r^*$ and for $1 \leqq i < m$, ρ_i is $f(z_i) = z_{i+1}$ or $f(z_{i+1}) = z_i$, and ρ_m is $f(z_m) = v$ or $f(v) = z_m$.

Define "v is above u" to mean that the (only) path from u to v contains $f(u)$. Note that aboveness is transitive, but there may be two different elements, each of which is above the other. Thus $\pi(u,v)$ says that there is a path (without repetitions) of length $\leqq r^*$ from u to v, and v is above u.

Let $n_0 \geqq 0$ be minimal such that $M \models (\neg \exists v)(\phi^*(v) \wedge \pi(f^{n_0}(a), v))$. The existence of n_0 is guaranteed by the hypothesis of Case 1. Let $c = f^{n_0}(a)$.

Assume $n_0 > 0$. Then by the minimality of n_0, $n_0 - 1$ will not work; that is, there exists $c_1 \in M'$ such that $M \models \phi^*(c_1) \wedge \pi(f^{n_0-1}(a), c_1)$. This path from $f^{n_0-1}(a)$ to c_1 contains $f(f^{n_0-1}(a)) = c$; by the conditions on c, c_1 is not above c. Thus there is $n_1 \geqq 0$ such that $f^{n_1}(c_1) = c$ and $f^m(a) \neq f^{m_1}(c_1)$ for all $m < n_0$, $m_1 < n_1$. (If $n_1 = 0$ then $c = c_1$ and the condition $f^m(a) \neq f^{m_1}(c_1)$ is vacuous.) See Figure 2.

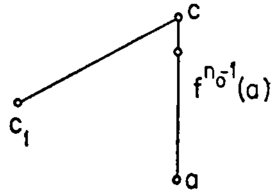

Fig. 2

Let $\tau(x) = \exists u \exists u_1(\phi^*(u) \wedge \phi^*(u_1) \wedge f^{n_0}(u) = x \wedge f^{n_1}(u_1) = x \wedge \bigwedge_{m<n_0, m_1<n_1} f^m(u) \neq f^{m_1}(u_1) \wedge (\neg \exists v)(\phi^*(v) \wedge \pi(x, v)))$.

We shall show that c is the only $x \in M'$ satisfying $M \models \tau(x)$. Clearly $M \models \tau(c)$. Now consider any $x \neq c$ in M'. By way of contradiction assume $M \models \tau(x)$, and let u, u_1 be as in τ. Either x is above c or c is above x; we get a contradiction in both cases.

Assume x is above c. Then either u or u_1 is above c, otherwise

$$\bigwedge_{m<n_0, m_1<n_1} f^m(u) \neq f^{m_1}(u_1)$$

would not hold. Assume without loss of generality that u is above c. Now $d(u, c) \leqq d(u, a) < r^*$, so that $M \models \pi(c, u)$, contradiction.

Now assume c is above x.

If also a is above x then $d(x, a) \leqq d(a, u) < r^*$ and hence $M \models \pi(x, a)$, contradiction.

If x is above a but a is not above x, that is, $f^t(a) = x$ for some $t > 0$, then

$$d(x, c_1) \leqq d(a, c_1) < r^* \text{ and hence } M \models \pi(x, c_1)$$

(since c_1 is also above x), contradiction. If $x = a$ the same holds.

Thus the only $x \in M'$ satisfying $M \models \tau(x)$ is c.

The only other elements of M which can satisfy τ are in the components of the b_i ($i = 1,\cdots,n$) and the same argument shows that they are at most one to a component. Thus there are finitely many in all of M and c is algebraic in M.

The case where $n_0 = 0$ is left to the reader.

Case 2. There is $x \in M'$ such that $M \vDash \phi^*(x)$ and $d(x,a) > r^*$. Say $d(x,a) = r$. Then $M \vDash \exists x(d(x,v) = r \wedge \phi^*(x))$ holds for $v = a$ and does not hold for $v = b_i$, $= 1,\cdots,n$. Put $\phi^{**}(v) = \phi^*(v) \wedge \exists x(d(x,v) = r \wedge \phi^*(x))$. Then $M \vDash \phi^{**}(v)$ holds for $v = a$ and the only such v in M is in M'.

M' is prime (by hypothesis of Theorem 3.5) so a realizes an atomic type in M'. Let $\psi(x)$ be a formula isolating the type of a in M'. Let $\psi'(x)$ be the corresponding formula from Corollary 2.2. We claim that $\phi^{**}(x) \wedge \psi'(x)$ isolates the type of a in M. Assume $M \vDash \phi^{**}(c) \wedge \psi'(c)$. Since $M \vDash \phi^{**}(c)$, $c \in M'$. Since $M \vDash \psi'(c)$, $M' \vDash \psi(c)$. Thus c and a realize the same type in M', and clearly c and a realize the same type in M. Thus a realizes an atomic type in M.

This completes the proof of Theorem 3.5 and hence of the Main Theorem.

4.

M. Rubin suggested the statement of the following theorem, which follows from Lemma 3.4, and was essentially proved above.

THEOREM 4.1. *M is minimal if and only if*

(i) *every component is minimal, and*

(ii) *for every component M', either M' contains an element algebraic in M, or there is a formula $\phi(x)$ such that $\emptyset \neq \{x : M \vDash \phi(x)\} \subseteq M'$.*

It is thus easy to see that if M is a minimal model of one function symbol, then there is no N such that $M \precneqq N$ and M and N realize the same types. See [5].

5.

The following example suggested by Gaifman illustrates some of the ideas used in the paper. Let M be a model of the form shown in Figure 3.

The conditions are:

(1) From c up, f is $1-1$.

(2) Each node branches at most into two.

(3) If a node branches into two then the two descending chains to the next branching point are finite and of unequal length.

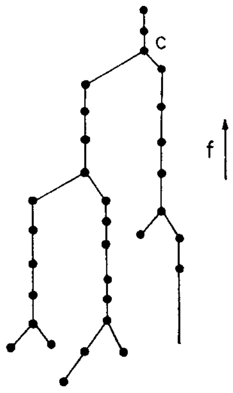

Fig. 3

(4) For every $r < \omega$ there is a node a, $a \neq c$, for which $Nbd^r a$ is isomorphic to $Nbd^r c$.

We leave it to the reader to show that M is minimal and not prime, and to find $2^{\aleph_0}$ non-isomorphic minimal models elementarily equivalent to M.

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